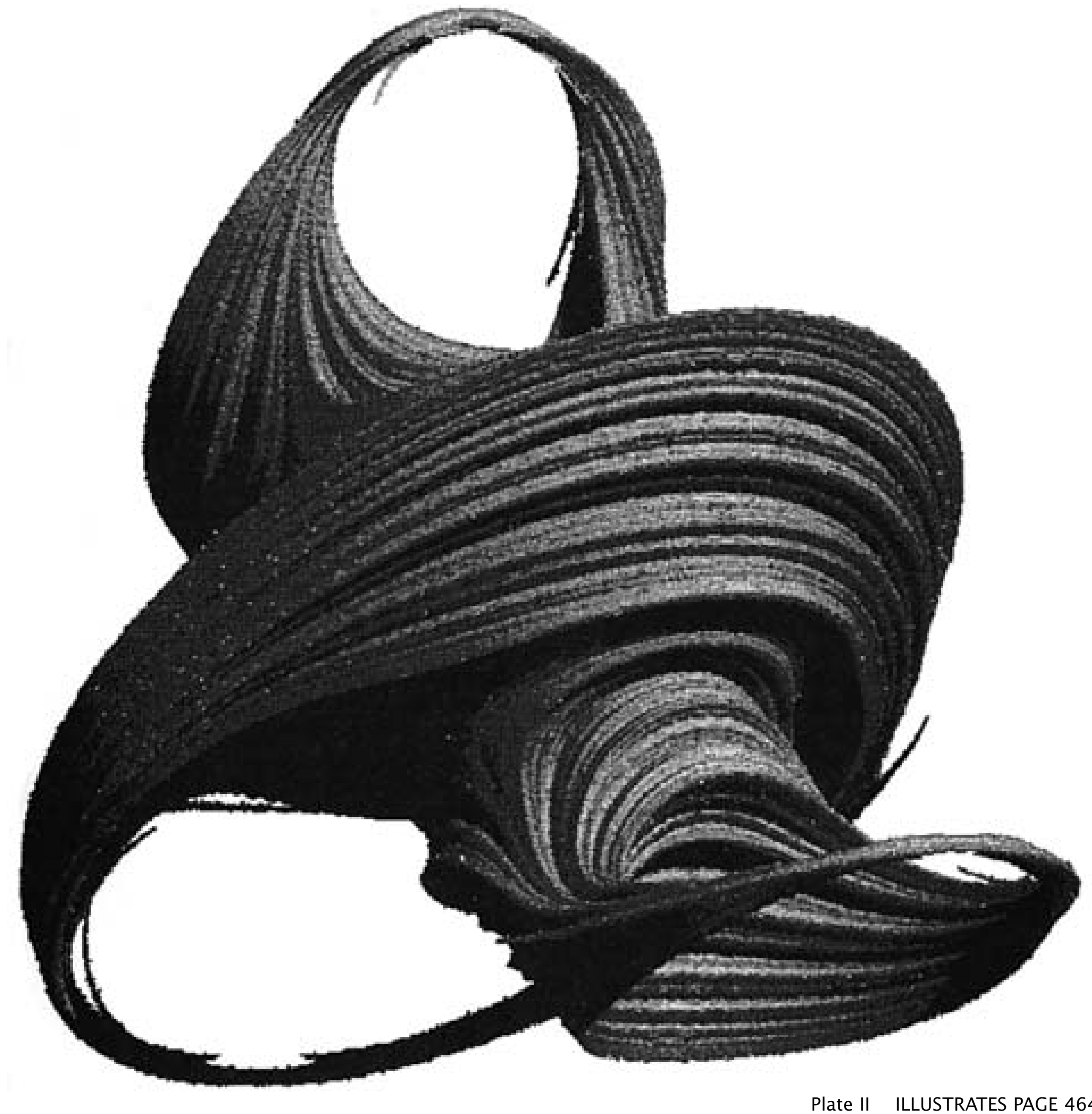
THE FRACTAL GEOMETRY OF NATURE



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THE FRACTAL GEOMETRY OF NATURE

Second Edition

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Peter L. Renz

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About the Author IBM Fellow at the IBM Thomas J. Watson Research Center. A Fellow of the American Academy of Arts and Sciences. Graduate, Ecole Polytechnique; M.S. and Ae.E. in Aeronautics, Caltech; Docteur ès Sciences Mathématiques, University of Paris, Dr. Mandelbrot's first positions were with the French Research Council (CNRS), The School of Mathematics and the Institute for Advanced Study (under J. von Neumann), and the University of Geneva. Immediately before joining IBM, he was a junior Professor of Applied Mathematics at the University of Lille and of Mathematical Analysis at Ecole Polytechnique. On leave from IBM, he had been a Visiting Professor of Economics, later of Applied Mathematics, and then of Mathematics at Harvard, of Engineering at Yale, of Phisiology at the Albert Einstein College of Medicine, and of Mathematics at the University of Paris-Sud. He has visited M.I.T. several times, first in the Electrical Engineering Department, and most recently as an Institute Lecturer. He had been a Fellow of the Guggenheim Foundation, Trumbull Lecturer at Yale, Samuel Wilks Lecturer at Princeton, Abraham Wald Lecturer at Columbia, Goodwin-Richards Lecturer at the University of Pennsylvania, and National Lecturer of Sigma Xi, the Scientific Research Society. Many times a Lecturer at College de France since 1973, he delivered there the leçons that eventually developed into the present Essay

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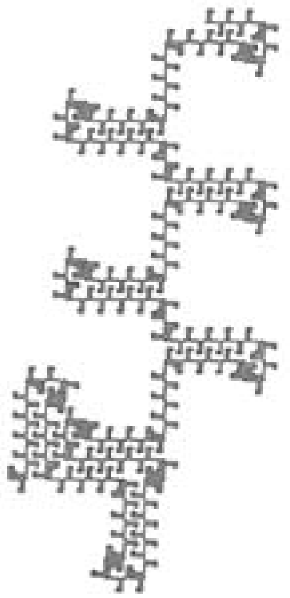

In Memoriam, B. et C.
Pour Aliette

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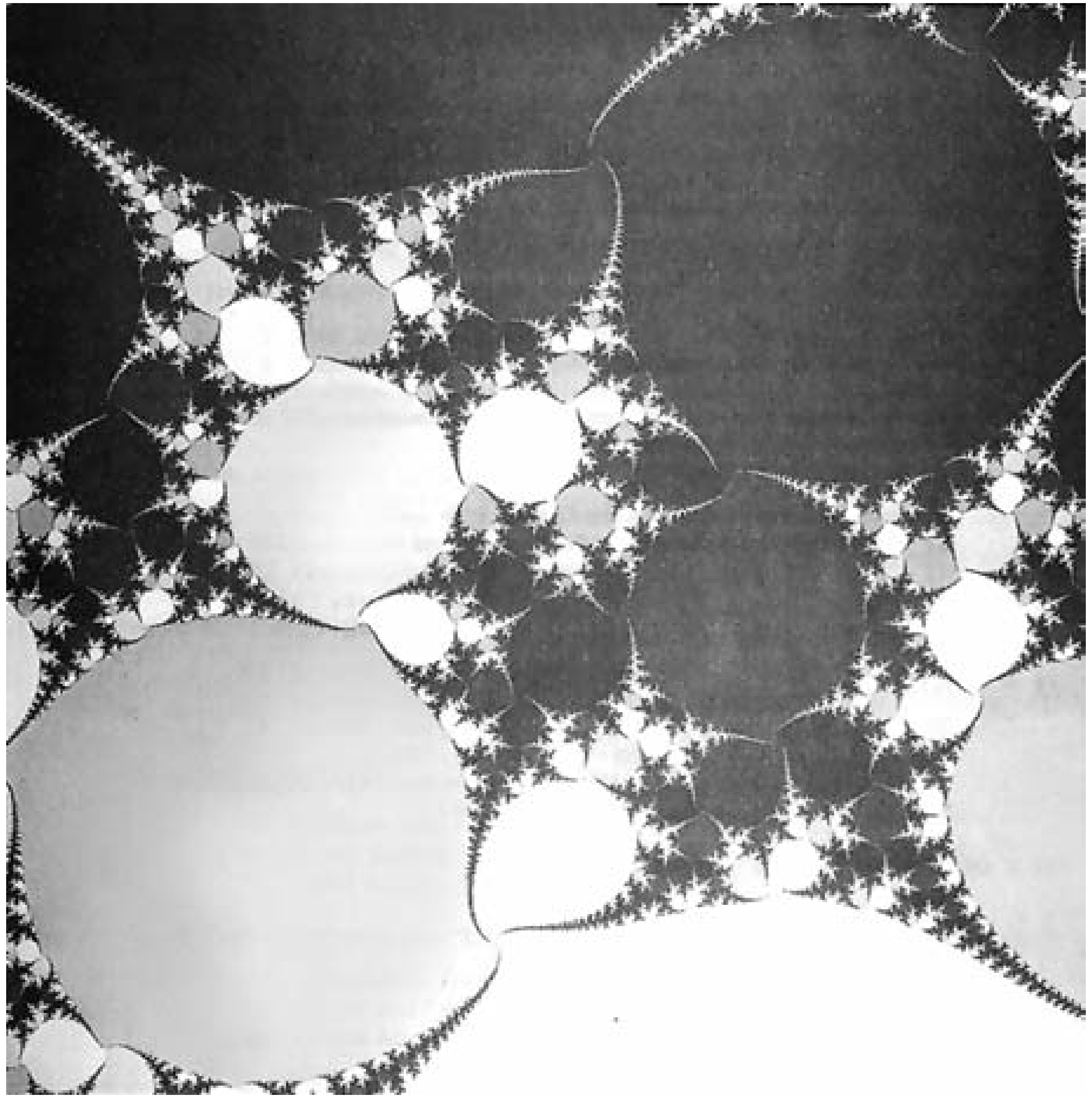

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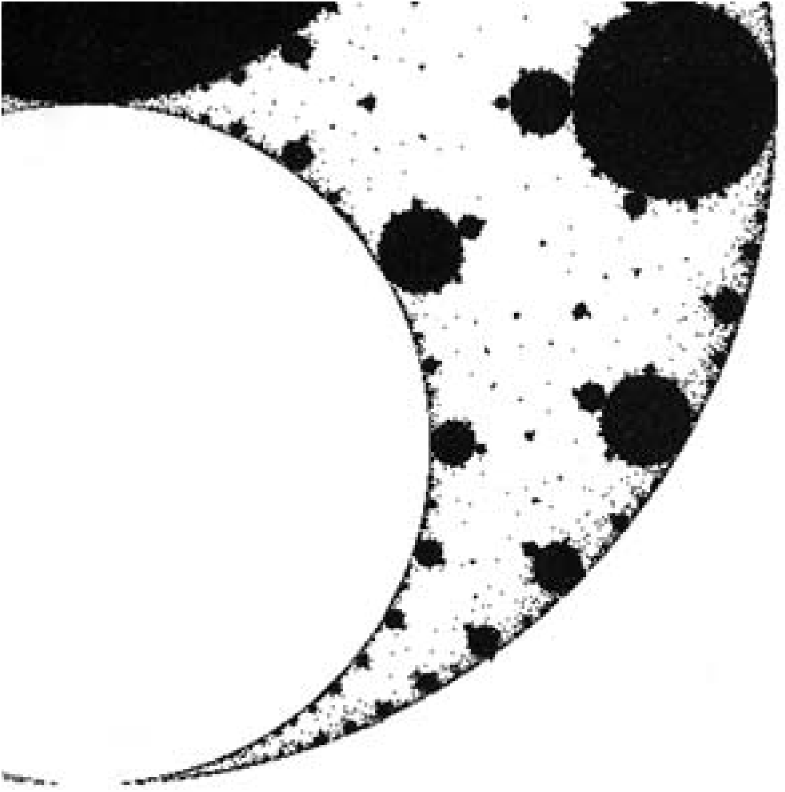
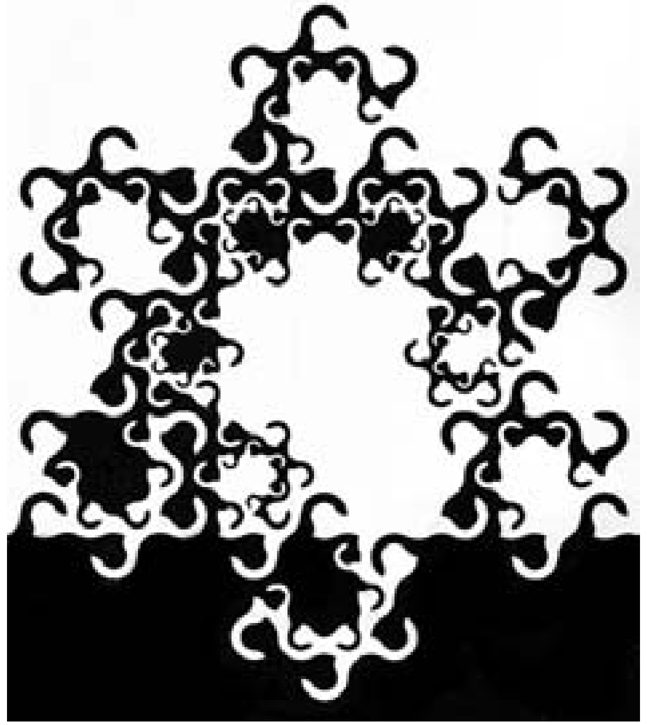
FOREWORD

This work follows and largely replaces my 1977 Essay, FRACTALS: FORM, CHANCE AND DIMENSION, which had followed and largely replaced my 1975 Essay in French, LES OBJETS FRACTALS: FORME, HASARD ET DIMENSION. Each stage involved new art, a few deletions, extensive rewriting that affected nearly every section, additions devoted to my older work, and—most important, extensive additions devoted to new developments.

My first scientific publication came out on April 30, 1951. Over the years, it had seemed to many that each of my investigations was aimed in a different direction. But this apparent disorder was misleading: it hid a strong unity of purpose, which the present Essay, like its two predecessors, is intended to reveal. Against odds, most of my works turn out to have been the birth pangs of a new scientific discipline.





1 Theme

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Plate 31 Monkeys Tree

Why is geometry often described as "cold" and "dry"? One reason lies in its inability to describe the shape of a cloud, a mountain, a coastline, or a tree. Clouds are not spheres, mountains are not cones, coastlines are not circles, and bark is not smooth, nor does lightning travel in a straight line.

More generally, I claim that many patterns of Nature are so irregular and fragmented, that, compared with Euclid—a term used in this work to denote all of standard geometry—Nature exhibits not simply a higher degree but an altogether different level of complexity. The number of distinct scales of length of natural patterns is for all practical purposes infinite.

Many of these illustrations are of shapes that had never been considered previously, but others represent known constructs, often for the first time. Indeed, while fractal geometry as such dates from 1975, many of its tools and concepts had been previously developed, for diverse purposes altogether different from mine. Through old stones inserted in the newly

PRESENTATION OF GOALS

This Essay brings together a number of analyses in diverse sciences, and it promotes a new mathematical and philosophical synthesis. Thus, it serves as both a *casebook* and a *manifesto.* Furthermore, it reveals a totally new world of plastic beauty.

One case study concerns the widely known application of widely known mathematics to a widely known natural problem: Wiener's geometric model of physical Brownian motion.

Now, the reason for bringing these prefaces together is that each helps one to understand the others because they share a common mathematical structure. F. J. Dyson has given an eloquent summary of these theme of mine.

"*Fractal* is a word invented by Mandelbrot to bring together under one heading a large class of objects that have [played]…an historical role…in the development of pure mathematics. A great revolution of ideas separates the classical mathematics of the 19th century from the modern mathematics of the 20th. Classical mathematics had its roots in the regular geometric structures of Euclid and the continuously evolving dynamics of Newton. Modern mathematics began with Cantor's set theory and Peano's space-filling curve. Historically, the revolution was forced by the discovery of mathematical structures that did not fit the patterns of Euclid and Newton. These new structures were regarded…as 'pathological,'…as a 'gallery of monsters,' kin to the cubist painting and atonal music that were upsetting established standards of taste in the arts at about the same time. The mathematicians who created the monsters regarded them as important in showing that the world of pure mathematics contains a richness of possibilities going far beyond the simple structures that they saw in Nature. Twentieth-century mathematics flowered in the belief that it had transcended completely the limitations imposed by its natural origins.

"Now, as Mandelbrot points out,…Nature has played a joke on the mathematicians. The 19th-century mathematicians may have been lacking in imagination, but Nature was not. The same pathological structures that the mathematicians invented to break loose from 19th-century naturalism turn out to be inherent in familiar objects all around us."*

In belief, I have confirmed Blaise Pascal's observation that imagination tires before Nature. ("L'imagination se lassera plutot de concevoir que la nature de fournir.")

RESTATEMENT OF GOALS

In sum, the present Essay describes the solutions I propose to a host of concrete problems, including very old ones, with the help of mathematics.

Scientists will (I am sure) be surprisedand delighted to find that not a few shapes they had to call *grainy, hydralike, in between, pimply, pocky,* and the like, can henceforth be approached in rigorous and vigorous quantitative fashion.

And the art can be enjoyed for itself. ■

*From "Characterizing Irregularity" by Freeman Dyson, Science, May 12, 1978, vol. 200, no. 4342, pp. 677–678. Copyright © 1978 by the American Association for the Advancement of Science.

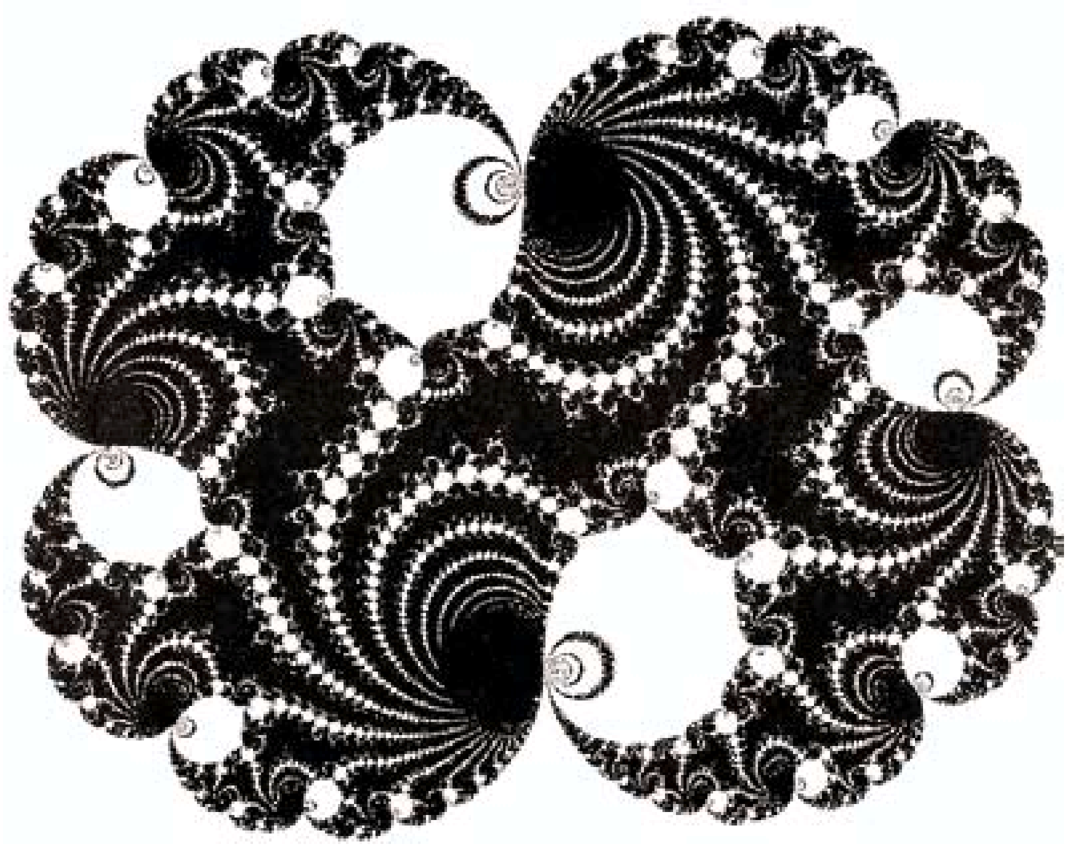
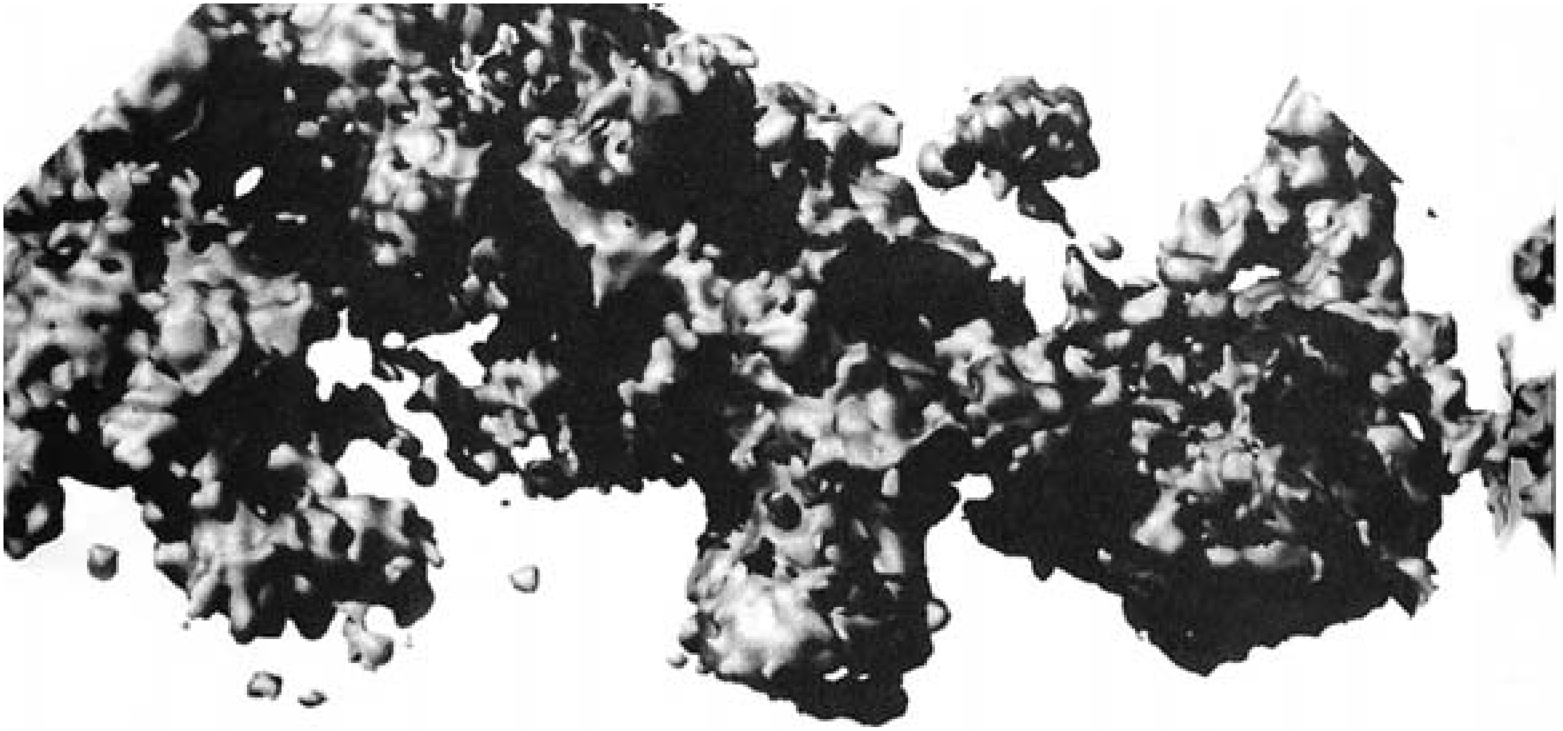


Plate 191



Plates 10 and 11 Artificial Fractal Flakes

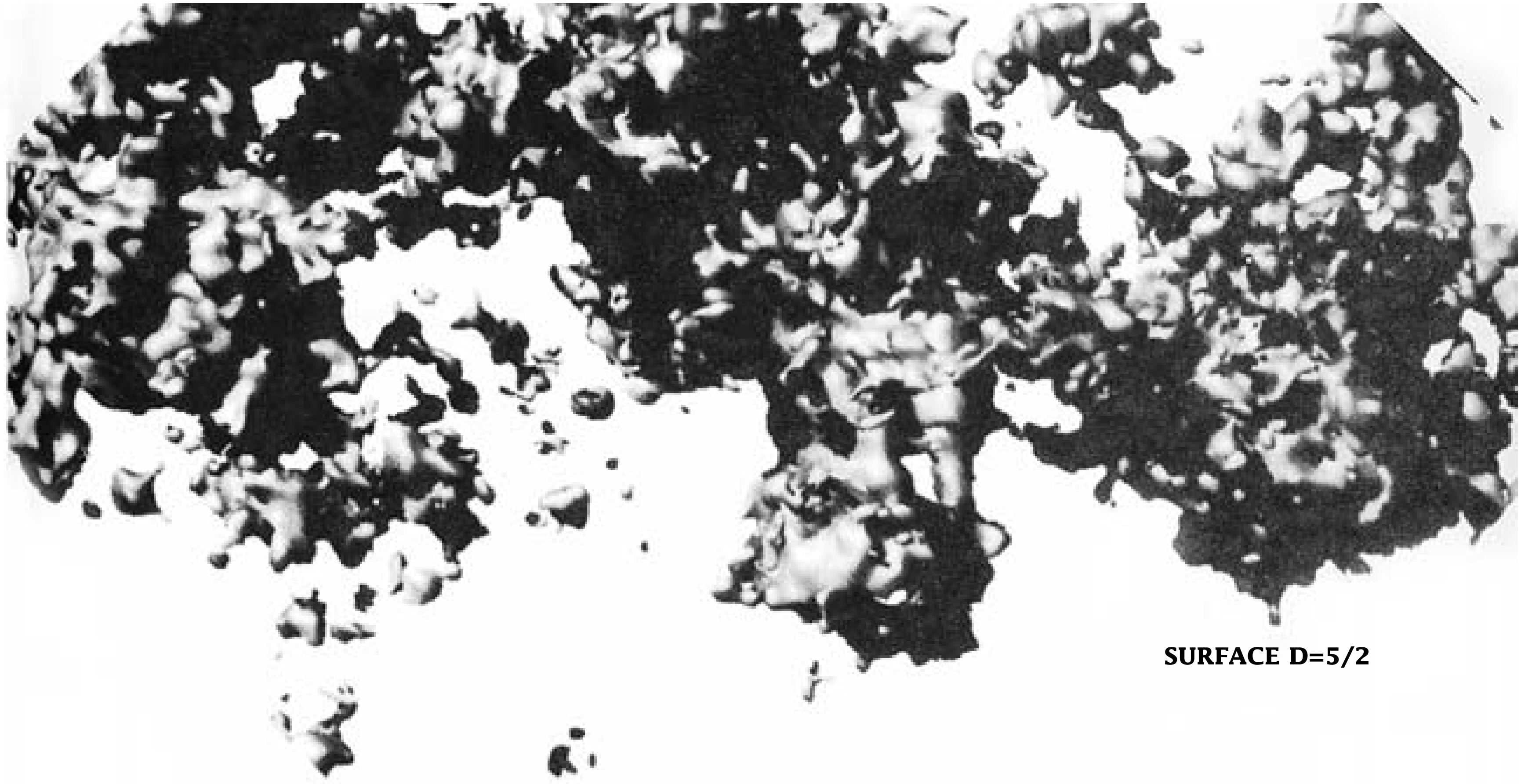
“Consider, for instance, the way in which we define the density of air at a given point and at a given moment. We picture a sphere of volume v centered at that point and including the mass m. The quotient m/v is the mean density within the sphere, and by true density we denote some limiting value of this quotient. This notion, however, implies that at the given moment the mean density is practically constant for spheres below a certain volume. This mean density may be notably different for spheres containing 1,000 cubic meters and 1 cubic centimeter respectively, but it is expected to vary only by 1 in 1,000,000 when comparing 1 cubic centimeter to one-thousandth of a cubic millimeter.

“Suppose the volume becomes continually smaller. Instead of becoming less and less important, these fluctuations come to increase. For scales at which the Brownian motion shows great activity, fluctuations may attain 1 part in 1,000 and they become of the order of 1 part in 5 when the radius of the hypothetical spherule becomes of the order of a hundredth of a micron.

In an inspiring text quoted in Chapter 2, Jean Perrin comments on the form of the “white flakes that are obtained by salting a solution of soap.” These illustrations are meant to accompany Perrin’s remarks.

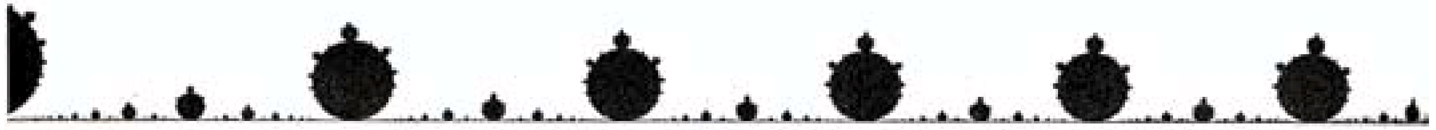
One must hasten to state that they are neither photographs nor computer reconstructions of any real object, be it a soap flake, a rain cloud, a volcanic cloud, a small asteroid, or a piece of virgin copper.

This and later digressions in this Essay are delimited by the new brackets ◄◁and ►. The latter is very bold, so as to be readily found by anyone who becomes lost in a digression and wants to skip ahead. But the “open bracket” symbol avoids attracting attention.



SURFACE D=5/2

SURFACE D=8/3



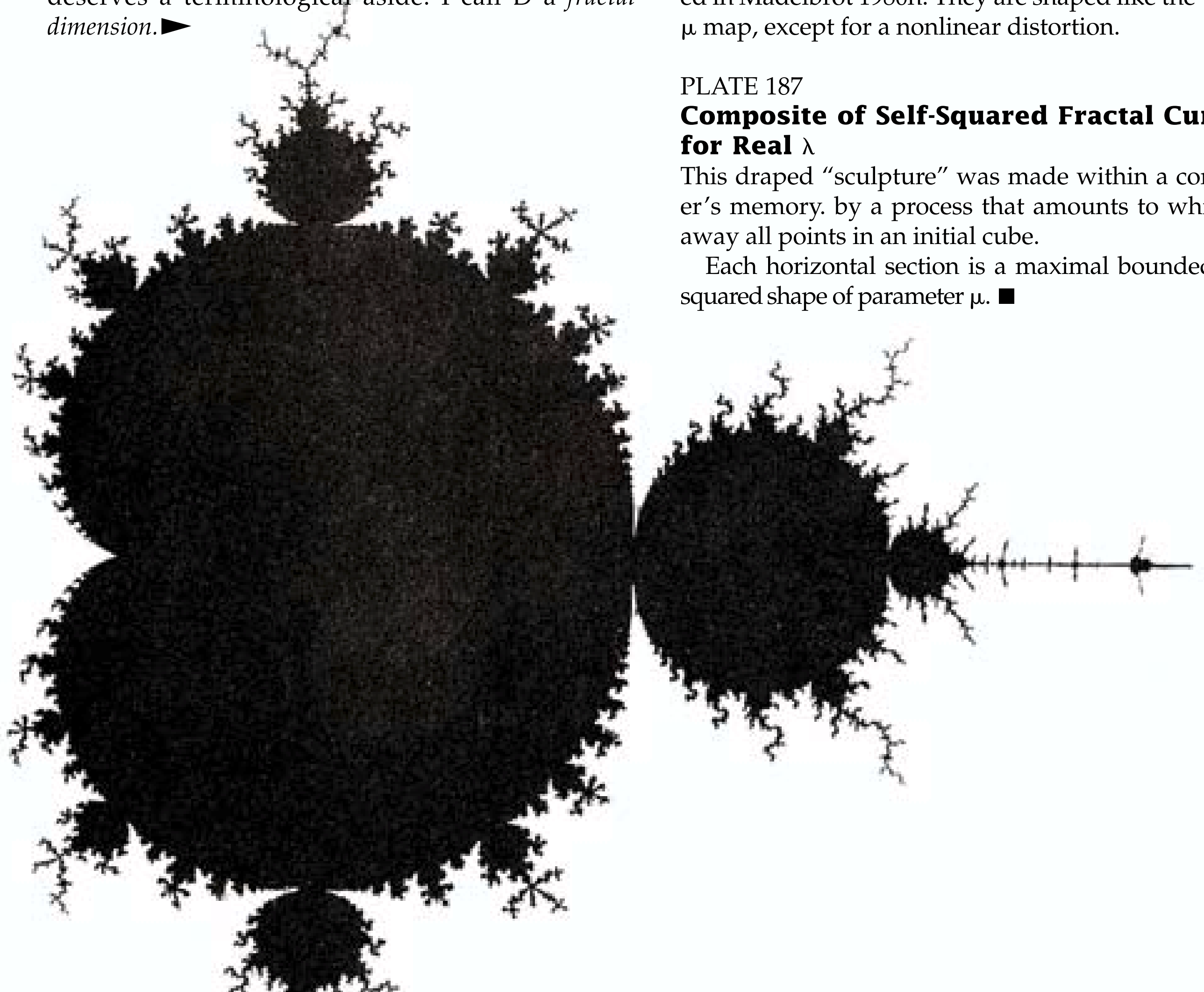
(Caption Continued From P. 188)

◄◁The fact that the basic frontals are dimensionally discordance can serve to transform the concept of fractal from an intuitive to a mathematical one.

$$D \geq DT_T$$

For all of Euclid, $D = D_T$. But nearly all sets in this Essay satisfy $D > D_T$.

◄◁ The striking fact that D need not be an integer deserves a terminological aside. I call D a *fractal dimension*.►



TOP PLATE 188. This is part of the inverse of the λ-map with respect to l = λ. Examining on the λ-map the sprouts whose roots are of the form $\lambda=\exp(2\pi/n)$, one gains the impression that "corresponding points" lie on circles. The present plate provides confirmation. Other perceived circles are confirmed by different inversions.

ISLAND MOLECULES. Many of the "spots" around the maps are genuine "island molecules," first reported in Madelbrot 1980n. They are shaped like the whole μ map, except for a nonlinear distortion.

PLATE 187

Composite of Self-Squared Fractal Curves for Real λ

This draped "sculpture" was made within a computer's memory. by a process that amounts to whittling away all points in an initial cube.

Each horizontal section is a maximal bounded self-squared shape of parameter μ. ■

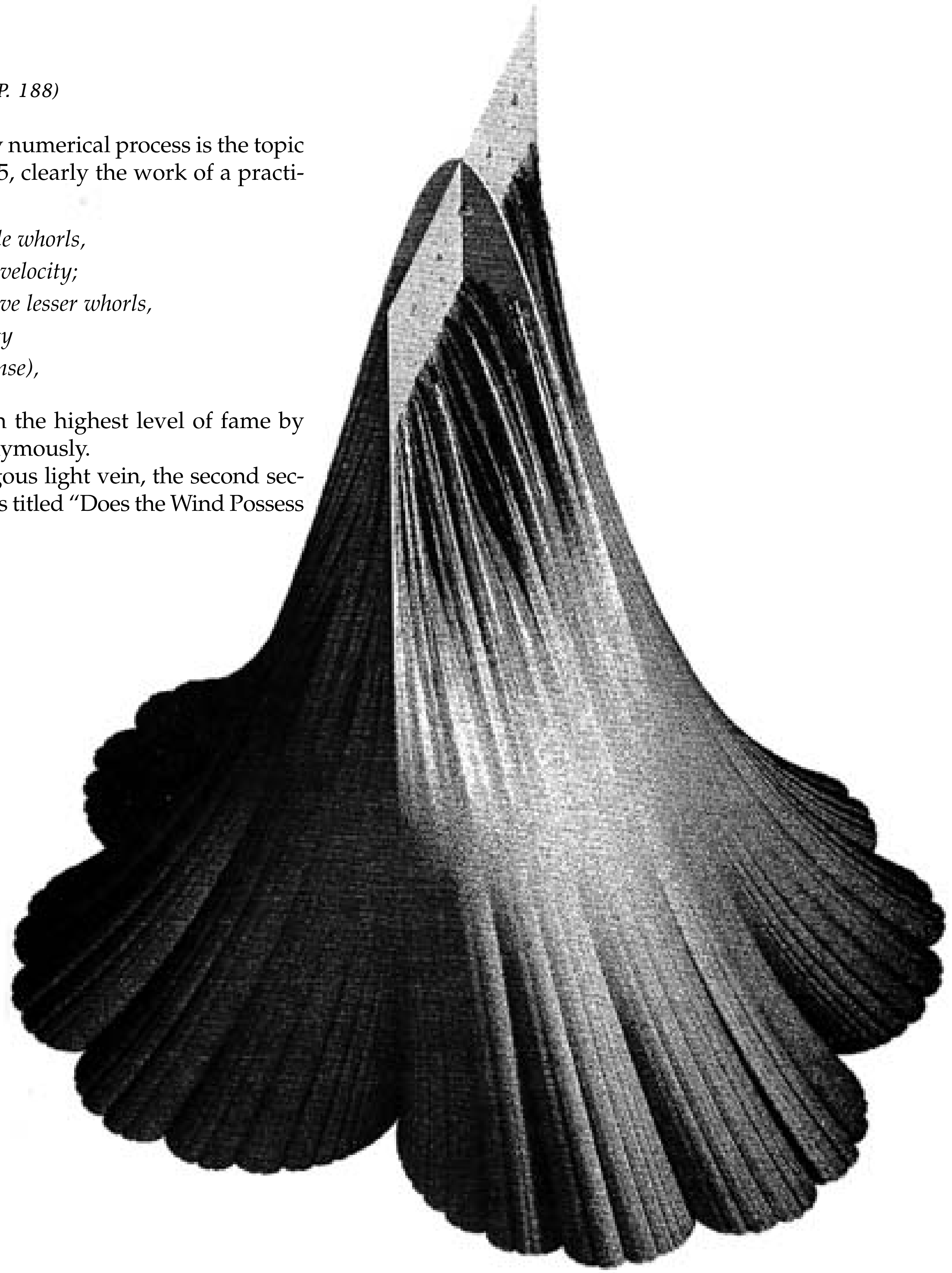
(Caption Continued From P. 188)

Weather prediction by numerical process is the topic of Richardson 1922–1965, clearly the work of a practical visionary.

Big whorls have little whorls,
Which feed on their velocity;
And little whorls have lesser whorls,
and so on to viscosity
(in the molecular sense),

In fact, these lines reach the highest level of fame by being often quoted anonymously.

In a somewhat analogous light vein, the second section of Richardson 1926 is titled "Does the Wind Possess a Velocity?"



39 Mathematical Backup and Addenda

Complicated formulas and mathematical definitions and references, avoided elsewhere, are brought together in this chapter, together with several mathematical and other addenda.

LIST OF ENTRIES

- Affinity and self-similarity (Self-)
- Brown fractal sets
- Dimension and covering of a set
- Dimension (Fourier) and heuristics
- Fractals (On the definition of)
- Hausdorff measure
  and Hausdorff Besicovitch dimension
- Indicator/coindicator functions
- Levy stable random variables and functions
- Lipschitz-Holder heuristics
- Music: Two properties of scaling
- Nonlacunar fractals
- Potentials & capacities.
- Frostman dimension
- Scaling under truncation
- Similarity dimension: its pitfalls
- Stationarity (Degrees of)
- Statistical analysis using R/S
- Weierstrass function and kin.
  ultraviolet and infrared catastrophes

AFFINITY AND SELF-SIMILARITY (SELF-)

In the text, the terms self-similar and self-affine (a neologism) are applied to either bounded or unbounded sets.

SELF-SIMILARITY

In the Euclidean space IRE, a real ratio $r > 0$ determines a transformation called *similarity.*

BOUNDED SETS. A bounded set S is self-similar, with respect to the ratio r and an integer N, when S is the union of N nonoverlapping subsets.

DEFINITION. The line-to-line Brownian function B(t) is a random function such that, for all t and Dt,

$$\Pr([B(t + \Delta t) - B(t)]/|\Delta t|^{H} < x) = \mathrm{erf}(x)$$

WHITE GAUSSIAN NOISE Representation. The function B(t) is continuous but nondifferentiable.

Acknowledgements

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Index of Names and Subjects

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ABBREVIATIONS HEAD

Abbreviations

New Features

[to come – new feature]

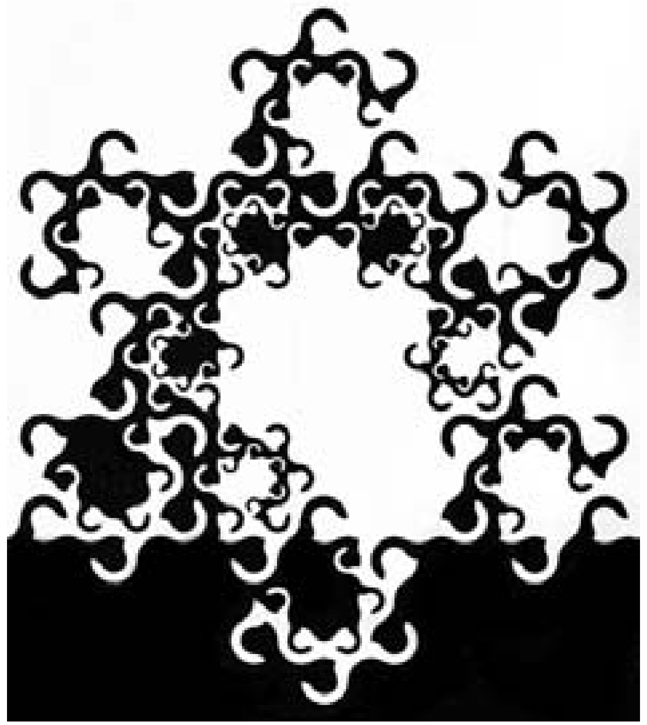

Plate 31 Monkeys Tree

13 ISLANDS, CLUSTERS, AND PERCOLATIONS; DIAMETER-NUMBER RELATIONS

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